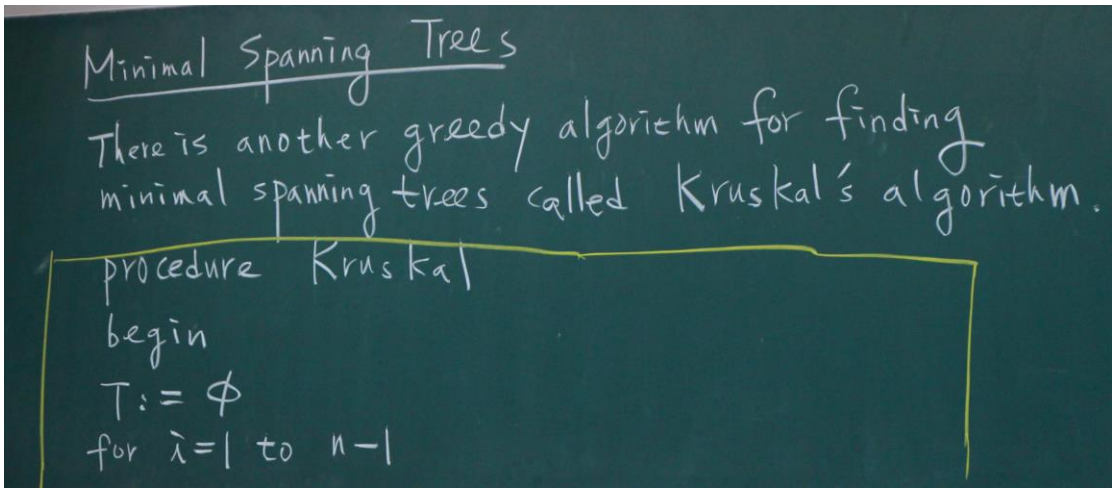
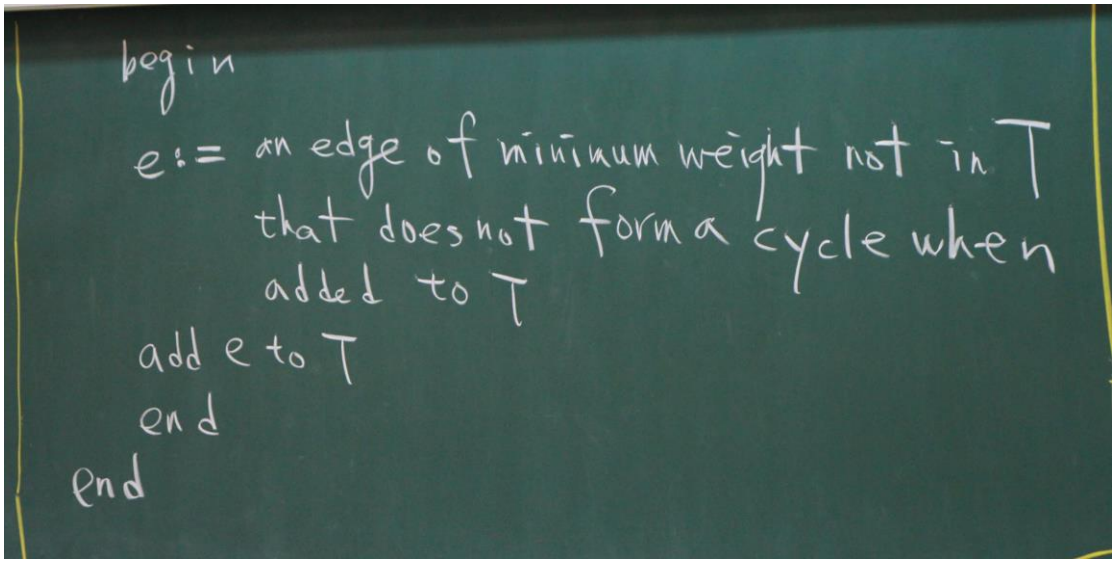
【10920 趙啟超教授離散數學 / 第 28 堂版書】





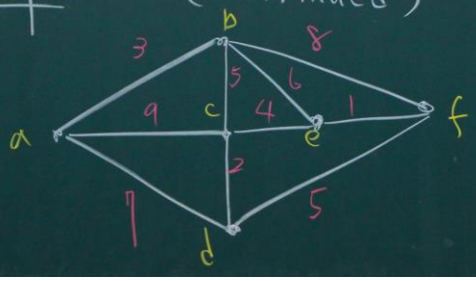
Minimal Spanning Trees

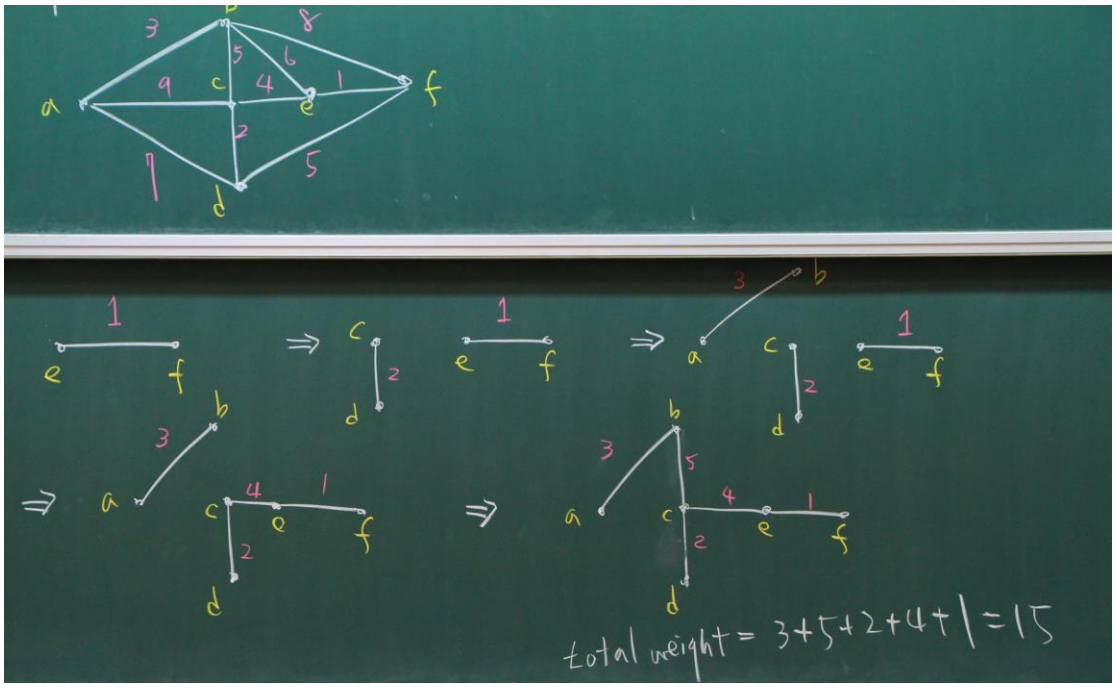
There is another greedy algorithm for finding minimal spanning trees called Kruskal's algorithm.

```
procedure Kruskal
begin
T := ∅
for i=1 to n-1
    begin
    e := an edge of minimum weight not in T
         that does not form a cycle when
         added to T
    add e to T
    end
end
```

We need to sort the edges by weight first and then examine them from smallest to largest.

Example (Continued)





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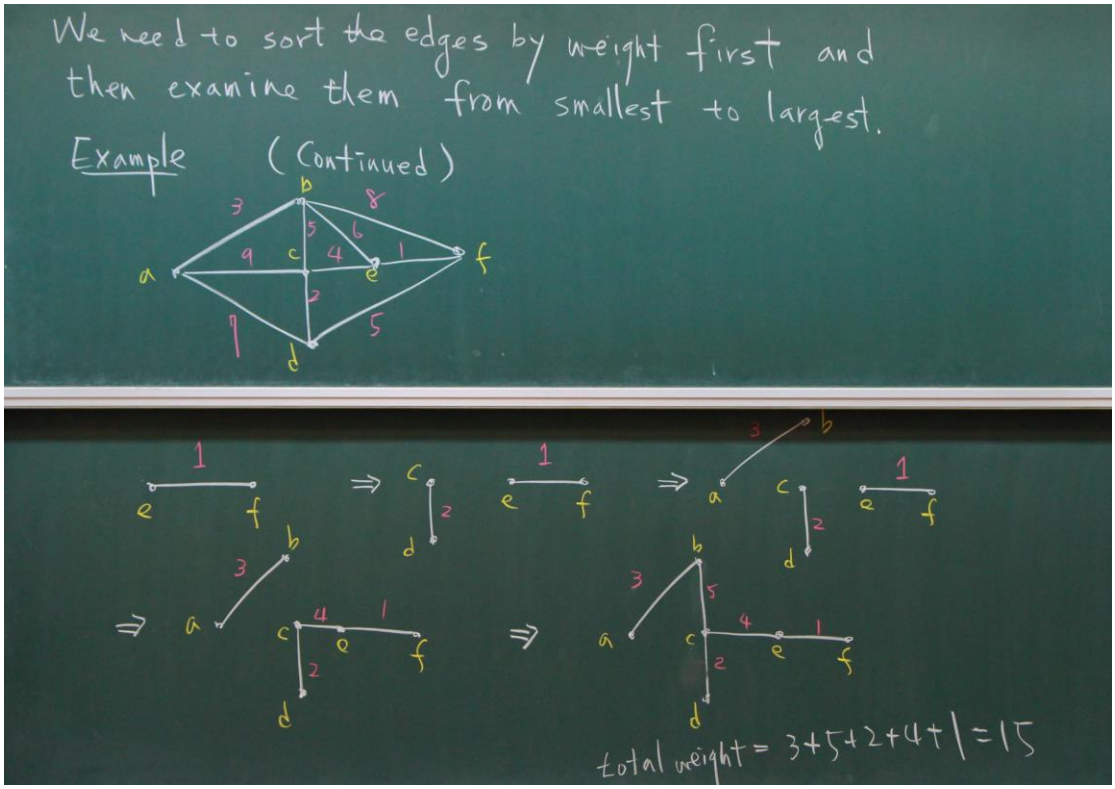

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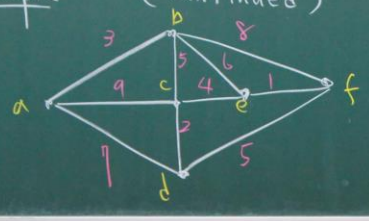

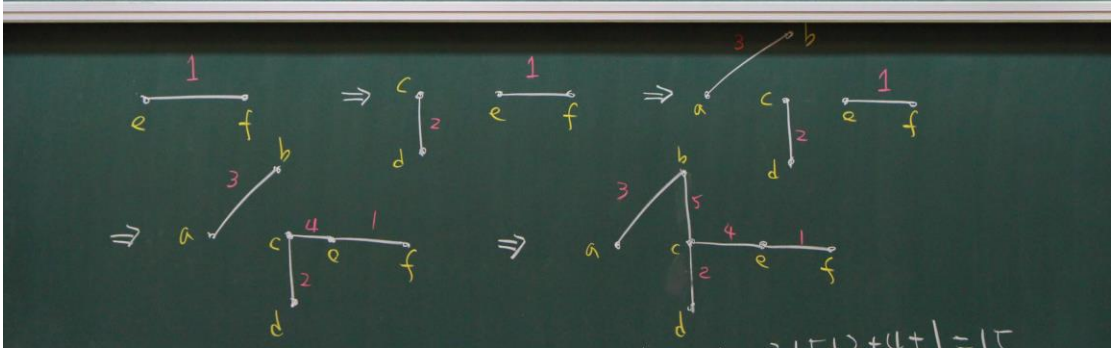

Total weight = 3+5+2+4+1 = 15

Proof of Correctness of Kruskal's Algorithm

The output from Kruskal's algorithm is a graph T, which is a subgraph of G, with n-1 edges and without cycles. We first show that T is connected. Suppose that T has more than one component. Since T has no cycles, each component of T is a tree which has one more vertex than edge.

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As T has n-1 edges, the number of vertices would be larger than n, contradicting the fact that T is a subgraph of G. Hence T has only one component, thus connected. Consequently, T is a tree with n-1 edges and n vertices, hence a spanning tree of G.

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We can then use a similar induction argument to the one used for Prim's algorithm. The induction hypothesis is that the graph T constructed so far is a subgraph of some minimal spanning tree M of G. This is certainly true at the start when T has no edges. Let e be the next edge added by the algorithm. Our goal is to show that $T \cup \{e\}$

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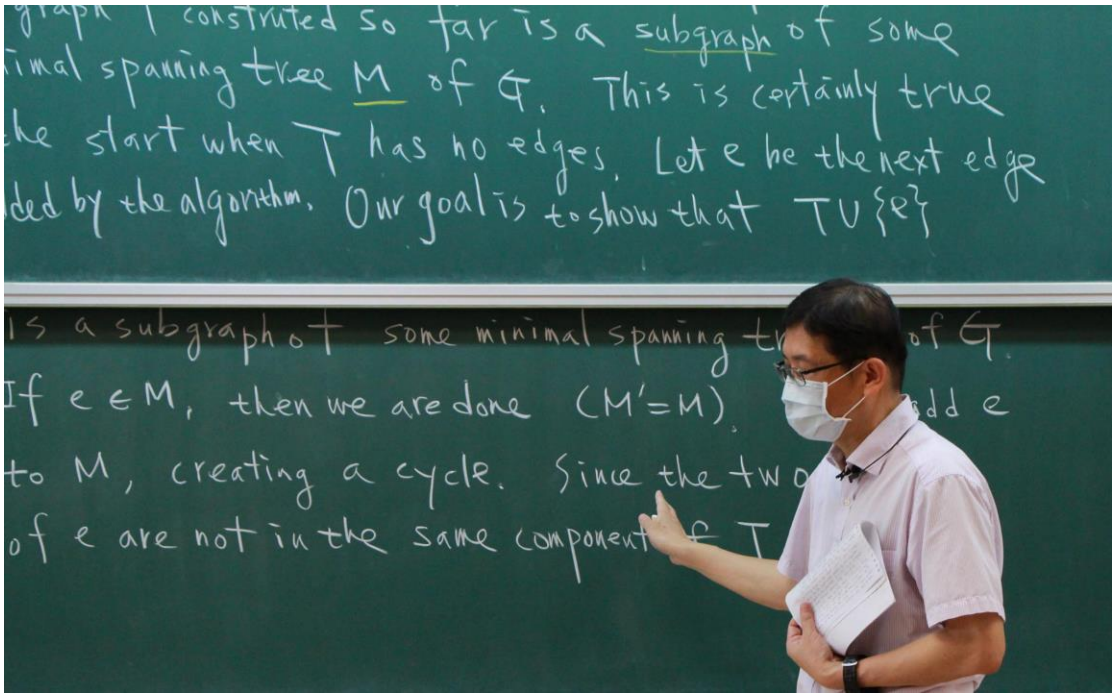

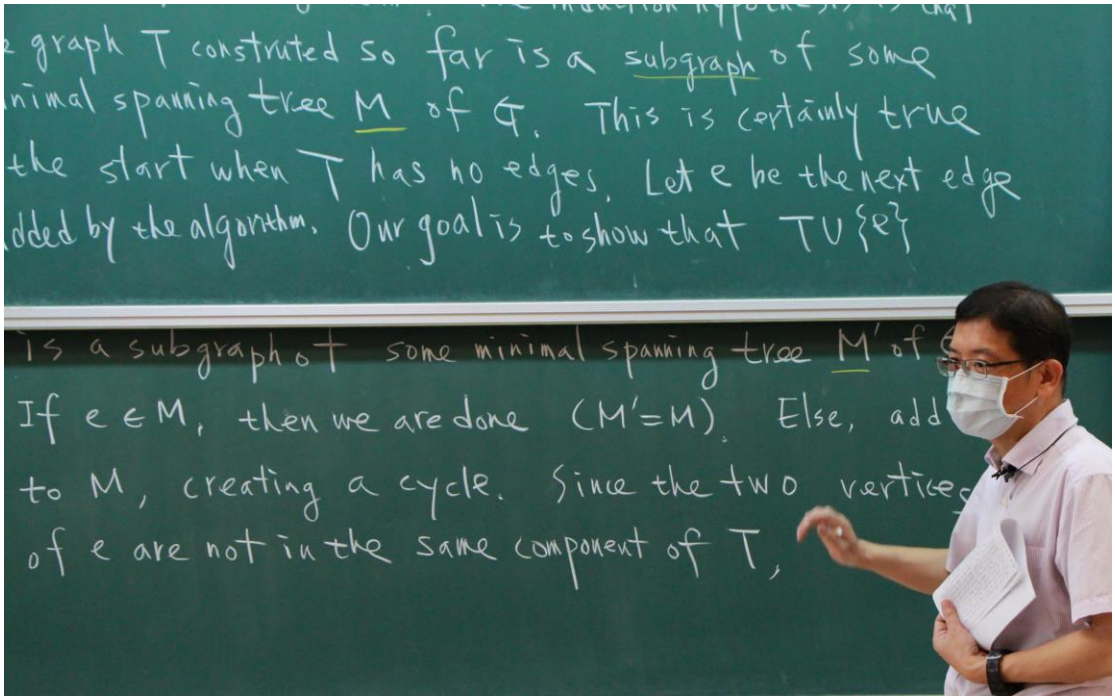

is a subgraph of some minimal spanning tree M' of G.
If $e \in M$, then we are done ($M' = M$). Else, add e
to M, creating a cycle. Since the two vertices
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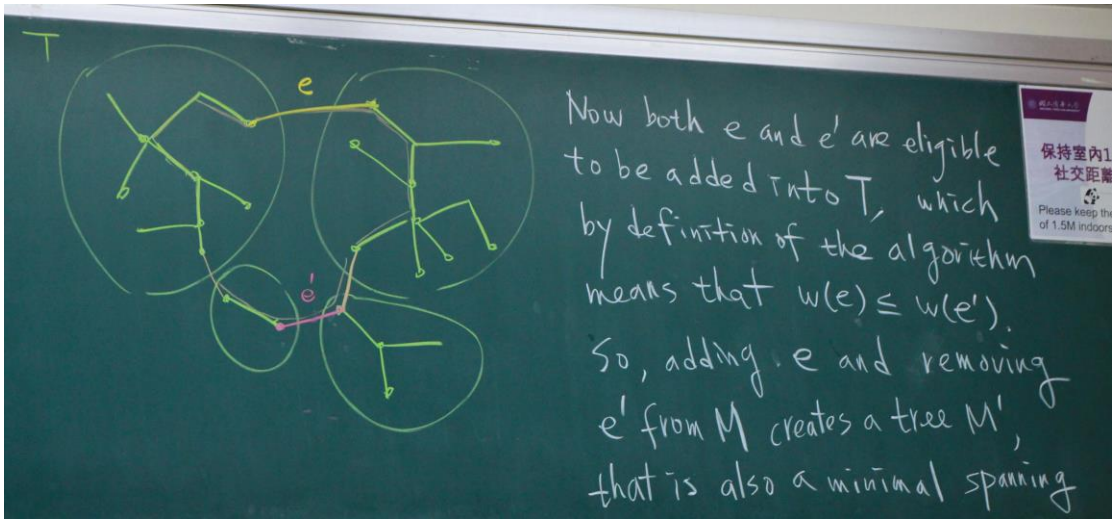
T
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e'
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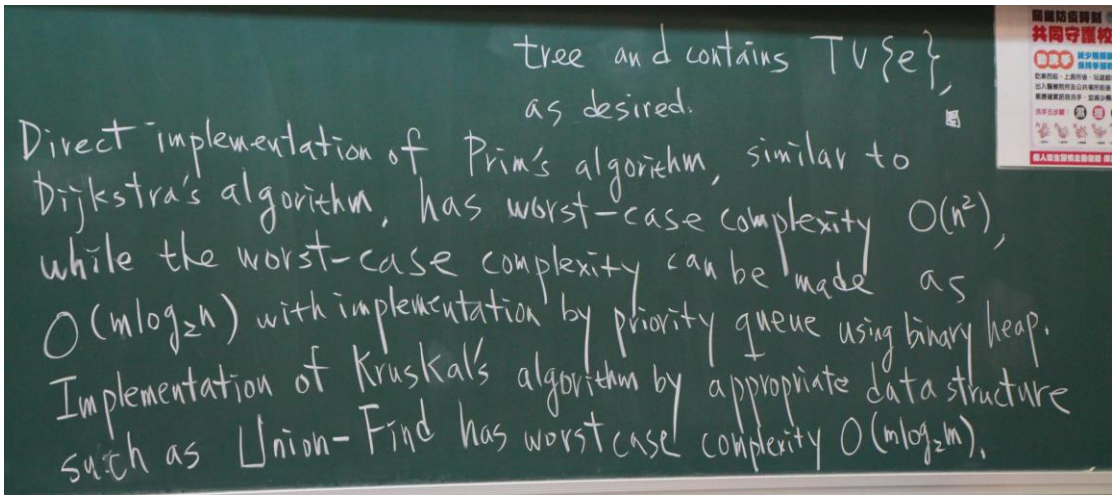
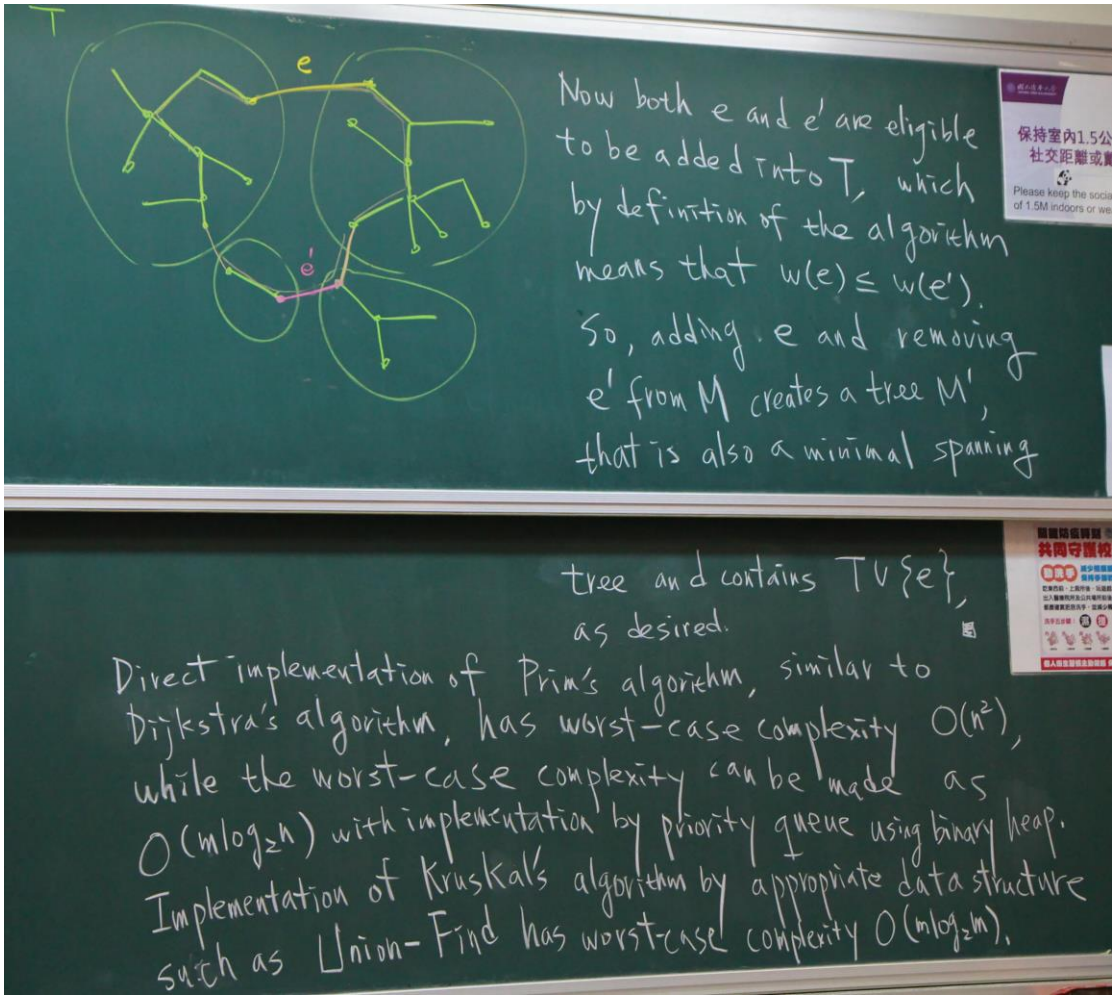

tree and contains T∪{e}, as desired.
Direct implementation of Prim's algorithm, similar to Dijkstra's algorithm, has worst-case complexity $O(n^2)$, while the worst-case complexity can be made as $O(m\log_2 n)$ with implementation by priority queue using binary heap.
Implementation of Kruskal's algorithm by appropriate data structure such as Union-Find has worst case complexity $O(m\log_2 m)$.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Now both e and e' are eligible to be added into T, which by definition of the algorithm means that w(e) ≤ w(e'). So, adding e and removing e' from M creates a tree M', that is also a minimal spanning tree and contains T ∪ {e}, as desired.
Direct implementation of Prim's algorithm, similar to Dijkstra's algorithm, has worst-case complexity O(n²), while the worst-case complexity can be made as O(m log₂ n) with implementation by priority queue using binary heap. Implementation of Kruskal's algorithm by appropriate data structure such as Union-Find has worst-case complexity O(m log₂ m).